ON WEIGHTED POINCARÉ INEQUALITIES

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ABSTRACT. The aim of this note is to show that Poincaré inequalities imply corresponding weighted versions in a quite general setting. The proof is short and does not involve covering arguments.

1. INTRODUCTION

Let (X, ρ) be a metric space with a positive σ-finite Borel measure dx, we will write $|E| = \int_E dx$ for the measure of a Borel set $E \subset X$. We fix some point $x_0 \in X$ and set $B_r = \{x \in X : \rho(x, x_0) < r\}$, $\overline{B}_r = \{x \in X : \rho(x, x_0) \le r\}$.

We call a function $\phi : B_1 \to [0, \infty)$ an *admissible weight*, if ϕ is a radial function, i.e. $\phi = \Phi(\rho(\cdot, x_0))$ and its profile Φ is nonincreasing and right-continuous with left-limits. We assume that ϕ is not identically zero on $B_1 \setminus \overline{B}_{1/2}$.

For any admissible weight ϕ there exists a positive, non-zero σ-finite Borel measure ν on $(\frac{1}{2}, 1]$, such that

$$\phi(x) = \int_{\rho(x,x_0)\vee 1/2}^{1} \nu(dt) = \int_{1/2}^{1} \chi_{B_t}(x)\, \nu(dt), \quad x \in B_1 \setminus \overline{B}_{1/2}. \tag{1}$$

(Note that we put $\int_a^b f(t)\, \nu(dt) := \int_{(a,b]} f(t)\, \nu(dt)$.)

For a function u we denote by

$$u_E = \frac{1}{|E|} \int_E u(x)\, dx$$

the mean of u over the set E, and by

$$u_E^{\phi} = \frac{\int_E u(x)\phi(x)\, dx}{\int_E \phi(x)\, dx}$$

the mean of u over the set $E \subset B_1$ with respect to the weight function ϕ.

Our main result is the following:

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Theorem 1. *Let $1 \le p < \infty$ and let ϕ be an admissible weight with $\phi = \Phi(\rho(\cdot, x_0))$. Let $F : L^p(X) \times (\frac{1}{2}, 1] \to [0, \infty]$ be a functional satisfying*

$$F(u + a, r) = F(u, r), \quad a \in \mathbb{R}, \tag{2}$$

$$\int_{B_r} |u(x) - u_{B_r}|^p \, dx \le F(u, r), \tag{3}$$

for every $r \in (\frac{1}{2}, 1]$ and every $u \in L^p(X)$. Then for $M = \frac{8^p |B_1|}{|B_{1/2}|} \frac{\Phi(0)}{\Phi(1/2)}$

$$\int_{B_1} |u(x) - u^{\phi}_{B_1}|^p \phi(x) \, dx \le M \int_{1/2}^{1} F(u, t) \, \nu(dt) \tag{4}$$

for every $u \in L^p(B_1)$, where ν is as in (1).

By choosing the functional F appropriately, (4) becomes a Poincaré inequality with weight ϕ, see Section 3. Such inequalities have been studied extensively because of their importance for the regularity theory of partial differential equations, see the exposition in [5].

2. Proof

Lemma 2. *Let Ω be a finite measure space and $p \ge 1$. Assume $f \in L^p(\Omega)$ with $\int_\Omega f = 0$. Then*

$$\|f + a\|_{L^p(\Omega)} \ge \tfrac{1}{2} \|f\|_{L^p(\Omega)}$$

for every $a \in \mathbb{R}$.

Proof. We may assume $a > 0$. Then

$$\int_{\Omega \cap \{f > 0\}} |f + a|^p \ge \int_{\Omega \cap \{f > 0\}} |f|^p \quad \text{and} \quad \int_{\Omega \cap \{f < -2a\}} |f + a|^p \ge 2^{-p} \int_{\Omega \cap \{f < -2a\}} |f|^p.$$

Furthermore, since $\int_{\Omega \cap \{f \le 0\}} |f| = \int_{\Omega \cap \{f > 0\}} |f|$, we obtain

$$\int_{\Omega \cap \{-2a \le f \le 0\}} |f|^p \le (2a)^{p-1} \int_{\Omega \cap \{-2a \le f \le 0\}} |f| \le (2a)^{p-1} \int_{\Omega \cap \{f > 0\}} |f| \le 2^{p-1} \int_{\Omega \cap \{f > 0\}} |f + a|^p ,$$

where we use $a^{p-1} b \le (b + a)^{p-1}(b + a)$ for positive a, b. Combining these observations we obtain the result. □

Proof of Theorem 1. First we observe that it is enough to prove that

$$\int_{B_1} |u(x) - u^{\tilde{\phi}}_{B_1}|^p \tilde{\phi}(x) \, dx \le \frac{2^{2p} |B_1|}{|B_{1/2}|} \int_{1/2}^{1} F(u, t) \, \nu(dt), \tag{5}$$

where $\tilde{\phi}(x) = \phi(x) \wedge \Phi(\frac{1}{2})$. Indeed, we have

$$\frac{\Phi(\frac{1}{2})}{\Phi(0)} \phi(x) \le \phi(x) \wedge \Phi(\tfrac{1}{2}) \le \phi(x).$$

Hence if (5) holds, then

$$\begin{aligned}\int_{B_1} |u(x)-u^{\tilde\phi}_{B_1}|^p \tilde\phi(x)\,dx &\geq \frac{\Phi(\frac12)}{\Phi(0)}\int_{B_1} |u(x)-u^{\tilde\phi}_{B_1}|^p \phi(x)\,dx\\ &\geq \frac{\Phi(\frac12)}{\Phi(0)}2^{-p}\int_{B_1} |u(x)-u^{\phi}_{B_1}|^p \phi(x)\,dx,\end{aligned}$$

where in the last line we have used Lemma 2.

Now we prove (5). To simplify the notation, we assume that $\phi(x)=\Phi(\frac12)$ for $x\in B_{1/2}$, so that $\tilde\phi=\phi$.

Because of (2), by subtracting a constant from u, we may and do assume that $u^{\phi}_{B_1}=0$, which means that

$$0=\int_{B_1} u(x)\phi(x)\,dx=\int_{1/2}^1\int_{B_t} u(x)\,dx\;\nu(dt)=\int_{1/2}^1 u_{B_t}|B_t|\;\nu(dt). \tag{6}$$

We start from the integral on the right hand side of (4) and use (3)

$$\begin{aligned}R&:=\int_{1/2}^1 F(u,t)\,\nu(dt)\geq\int_{1/2}^1\int_{B_t}|u(x)-u_{B_t}|^p\,dx\,\nu(dt)\\ &=\frac12\int_{1/2}^1\int_{B_t}|u(x)-u_{B_t}|^p\,dx\,\nu(dt)+\frac12\int_{B_1}\int_{1/2}^1|u(x)-u_{B_t}|^p\chi_{B_t}(x)\,\nu(dt)\,dx\\ &=:I_1+I_2\end{aligned}$$

(In fact $I_1=I_2$, but we treat them differently.) We now deal with the inner integral in I_2. For $x\in B_{1/2}$ we have

$$\int_{1/2}^1|u(x)-u_{B_t}|^p\chi_{B_t}(x)\,\nu(dt)\geq\frac{1}{|B_1|}\int_{1/2}^1|u(x)-u_{B_t}|^p|B_t|\,\nu(dt).$$

Since $\int_{1/2}^1 u_{B_t}|B_t|\,\nu(dt)=0$, by Lemma 2 we obtain

$$\int_{1/2}^1|u(x)-u_{B_t}|^p|B_t|\,\nu(dt)\geq 2^{-p}\int_{1/2}^1|u_{B_t}|^p|B_t|\,\nu(dt).$$

Therefore

$$I_2\geq\frac{2^{-p}}{2|B_1|}\int_{B_{1/2}}\int_{1/2}^1|u_{B_t}|^p|B_t|\,\nu(dt)\,dx=\frac{2^{-p}|B_{1/2}|}{2|B_1|}\int_{1/2}^1|u_{B_t}|^p|B_t|\,\nu(dt).$$

Using inequality $|a|^p+|b|^p\geq 2^{1-p}|a+b|^p$ we obtain

$$\begin{aligned}I_1+I_2&\geq\frac12\int_{1/2}^1\int_{B_t}\left(|u(x)-u_{B_t}|^p+\frac{2^{-p}|B_{1/2}|}{|B_1|}|u_{B_t}|^p\right)\,dx\,\nu(dt)\\ &\geq\frac{2^{-p}|B_{1/2}|}{2|B_1|}2^{1-p}\int_{1/2}^1\int_{B_t}|u(x)|^p\,dx\,\nu(dt)\\ &=\frac{|B_{1/2}|}{|B_1|}2^{-2p}\int_{B_1}|u(x)|^p\phi(x)\,dx\end{aligned}$$

and the proof is finished. □

3. Applications

Let us discuss some corollaries. Corollary 3 is well-known [5], although our class of admissible weights is more general. Proposition 4 allows to deduce a weighted Poincaré inequality for fractional Sobolev norms from an unweighted version. Corollaries 5 and 6 give a more concrete result for fractional Sobolev norms. The first allows for more general kernels and exponents p. Corollary 6 improves [2, Theorem 5.1] because the result is robust for $s \to 1-$ and allows for general weights and exponents p.

Corollary 3. *Let $p \geq 1$ and ϕ be an admissible weight. Consider $X = \mathbb{R}^d$ equipped with the Lebesgue measure and the Euclidean metric. There exists a positive constant C depending on p, d and ϕ such that*

$$\int_{B_1} |u(x) - u^{\phi}_{B_1}|^p \phi(x)\, dx \leq C \int_{B_1} |\nabla u(x)|^p \phi(x)\, dx, \tag{7}$$

for every $u \in W^{1,p}(B_1)$.

Proposition 4. *Let $p \geq 1$ and let ϕ be an admissible weight of the form $\phi = \Phi(\rho(\cdot, x_0))$. Assume that for some kernel $k : B_1 \times B_1 \to [0, \infty)$ and some positive constant C the following inequality holds*

$$\int_{B_r} |u(x) - u_{B_r}|^p\, dx \leq C \int_{B_r}\int_{B_r} |u(x) - u(y)|^p k(x,y)\, dy\, dx, \tag{8}$$

whenever $r \in (\frac{1}{2}, 1]$ and $u \in L^p(X)$. Then with $M = \frac{8^p |B_1|}{|B_{1/2}|} \frac{\Phi(0)}{\Phi(1/2)}$

$$\int_{B_1} |u(x) - u^{\phi}_{B_1}|^p \phi(x)\, dx \leq CM \int_{B_1}\int_{B_1} |u(x) - u(y)|^p k(x,y)(\phi(y) \wedge \phi(x))\, dy\, dx \tag{9}$$

for $u \in L^p(X)$.

Corollary 5. *Let ϕ be an admissible weight of the form $\phi = \Phi(\rho(\cdot, x_0))$ and $p \geq 1$. Let $k : B_1 \times B_1 \to [0, \infty)$ be a kernel satisfying $k \geq c$ for some constant $c > 0$. There is a positive constant M depending on d, p and Φ such that for $u \in L^p(X)$*

$$\int_{B_1} |u(x) - u^{\phi}_{B_1}|^p \phi(x)\, dx \leq \frac{M}{c} \int_{B_1}\int_{B_1} |u(x) - u(y)|^p k(x,y)(\phi(y) \wedge \phi(x))\, dy\, dx \tag{10}$$

for $u \in L^p(X)$.

Corollary 6. *Let $p \geq 1$, $r \geq 1$ and $0 < s_0 \leq s < 1$. Consider $X = \mathbb{R}^d$ equipped with the Lebesgue measure and the Euclidean metric. Let ϕ be an admissible weight of the form $\phi = \Phi(|\cdot|)$. Then there exists a positive constant C depending on p, d, s_0 and Φ such that*

$$\int_{B_1} |u(x) - u^{\phi}_{B_1}|^p \phi(x)\, dx \leq C(1-s)\, r^{p(1-s)} \int_{B_1}\int_{B_1} \frac{|u(x) - u(y)|^p}{|x-y|^{d+ps}} \chi_{\{|x-y| \leq \frac{1}{r}\}} (\phi(y) \wedge \phi(x))\, dy\, dx \tag{11}$$

for all $u \in L^p(B_1)$.

Proof of Corollary 3. It is well-known that the following Poincaré inequality holds

$$\int_{B_r} |u(x) - u_{B_r}|^p\, dx \leq c\, r^p \int_{B_r} |\nabla u(x)|^p\, dx \tag{12}$$

for every $u \in W^{1,p}(B_r)$ and $r > 0$ where $c > 0$ depends on p and d. Set

$$F(u,r) = c\, r^p \int_{B_r} |\nabla u(x)|^p \, dx,$$

for $u \in W^{1,p}(B_1)$ and $F(u,r) = \infty$ otherwise. Then for $u \in W^{1,p}(B_1)$

$$\begin{aligned}\int_{1/2}^{1} F(u,t)\,\nu(dt) &= c\int_{1/2}^{1} t^p \int_{B_1} |\nabla u(x)|^p \chi_{B_t}(x)\,dx\,\nu(dt)\\ &\le c\int_{B_1} |\nabla u(x)|^p \int_{1/2}^{1} \chi_{B_t}(x)\,\nu(dt)\,dx = c\int_{B_1} |\nabla u(x)|^p \phi(x)\,dx.\end{aligned}$$

By Theorem 1 the assertion follows with $C = 2^{3p+d}\frac{\Phi(0)}{\Phi(1/2)}c$. □

Proof of Proposition 4. Let

$$F(u,r) = C\int_{B_r}\int_{B_r} |u(x)-u(y)|^p k(x,y)\,dy\,dx.$$

Then

$$\begin{aligned}\int_{1/2}^{1} F(u,t)\,\nu(dt) &= C\int_{1/2}^{1}\int_{B_1}\int_{B_1} |u(x)-u(y)|^p k(x,y)\chi_{B_t}(y)\chi_{B_t}(x)\,dy\,dx\,\nu(dt)\\ &= C\int_{B_1}\int_{B_1} |u(x)-u(y)|^p k(x,y)\int_{1/2}^{1}\chi_{B_t}(y)\chi_{B_t}(x)\,\nu(dt)\,dy\,dx\\ &= C\int_{B_1}\int_{B_1} |u(x)-u(y)|^p k(x,y)(\phi(y)\wedge\phi(x))\,dy\,dx.\end{aligned}$$

The assertion now follows from Theorem 1. □

Lemma 7. *Let $r \ge 1$, $p \ge 1$ and $0 < s < 1$. Then*

$$\int_{B_1}\int_{B_1} \frac{|u(x)-u(y)|^p}{|x-y|^{d+ps}}\,dy\,dx \le (3r)^{p(1-s)}\int_{B_1}\int_{B_1} \frac{|u(x)-u(y)|^p}{|x-y|^{d+ps}}\chi_{\{|x-y|\le\frac{1}{r}\}}\,dy\,dx \tag{13}$$

for all $u \in L^p(B_1)$.

Proof. Let n be a natural number such that $n \ge 2r > n-1$. We introduce

$$A_k = A_k(x,y) = \tfrac{k}{n}y + \tfrac{n-k}{n}x, \quad k = 0,1,\dots n.$$

Then

$$\begin{aligned}I = \int_{B_1}\int_{B_1} \frac{|u(x)-u(y)|^p}{|x-y|^{d+ps}}\,dy\,dx &= \int_{B_1}\int_{B_1} \frac{|\sum_{k=1}^{n}(u(A_{k-1})-u(A_k))|^p}{|x-y|^{d+ps}}\,dy\,dx\\ \le n^{p-1}\sum_{k=1}^{n}\int_{B_1}\int_{B_1} \frac{|u(A_{k-1})-u(A_k)|^p}{|x-y|^{d+ps}}\,dy\,dx.\end{aligned}$$

Note that $|A_{k-1}-A_k| = \frac{1}{n}|x-y|$. If we substitute $\tilde{x} = A_{k-1}$, $\tilde{y} = A_k$, then $d\tilde{y}\,d\tilde{x} = n^{-d}dy\,dx$ (which follows by an elementary calculation, see also [3, page 570]). Moreover, $\tilde{x}$, $\tilde{y} \in B_1$

with $|\tilde{x} - \tilde{y}| \leq \frac{2}{n} \leq \frac{1}{r}$. Hence

$$I \leq n^{p-ps} \int_{B_1} \int_{B_1} \frac{|u(\tilde{x}) - u(\tilde{y})|^p}{|\tilde{x} - \tilde{y}|^{d+ps}} \chi_{\{|\tilde{x}-\tilde{y}| \leq \frac{1}{r}\}} \, d\tilde{y} \, d\tilde{x}.$$

Since $n < 2r + 1 \leq 3r$, the assertion follows. □

Proof of Corollary 5. First we use a well-known argument to obtain a nonweighted Poincaré inequality. By the convexity of the function $x \mapsto |x|^p$ it holds $|a+b|^p \geq |a|^p + bp|a|^{p-1} \operatorname{sgn}(a)$. Thus

$$\begin{aligned} \int_{B_r} \int_{B_r} |u(x) - u(y)|^p k(x,y) \, dy \, dx &\geq c \int_{B_r} \int_{B_r} |(u(x) - u_{B_r}) + (u_{B_r} - u(y))|^p \, dy \, dx \\ &\geq c|B_r| \int_{B_r} |u(x) - u_{B_r}|^p \, dx \\ &\geq c|B_{1/2}| \int_{B_r} |u(x) - u_{B_r}|^p \, dx, \end{aligned}$$

whenever $u \in L^p(B_r)$ and $\frac{1}{2} < r \leq 1$.

The assertion follows now from Proposition 4. □

Proof of Corollary 6. From [4] and [1, page 80] we know that there exists a constant $C = C(p, d, s_0)$, such that for $s_0 \leq s < 1$

$$\int_{B_r} |u(x) - u_{B_r}|^p \, dx \leq C(1-s) r^{ps} \int_{B_r} \int_{B_r} \frac{|u(x) - u(y)|^p}{|x-y|^{d+ps}} \, dy \, dx, \tag{14}$$

for all $u \in L^p(B_1)$. The assertion follows now from (14), Proposition 4 and Lemma 7. □

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